Resit Exam of Advanced Algebraic Structures
Block 1B, 2024–2025
April 9, 2025, 11:45 – 13:45
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Att	Q1	Q2	Q3	Q4	TOTAL
4					
4 pts	14 pts	4 pts	4 pts	14 pts	40 pts

Full Name: .... **Student Number:** ....

INSTRUCTIONS

- You have 2 hours to complete the exam.
- Write your name and student number on every page you hand in.
- You have to give complete arguments for all your answers.
- No electronic devices are allowed.
- You may use results obtained in the lecture, tutorial and homework problems unless it is explicitly asked to prove such a result.
- In total you can obtain at most 36 points on this exam. Your grade for the exam is $(P + 4)/4$, where P is the number of points you obtain on the exam.
- Good luck!

1. Let E be the splitting field of $x^{19} - 2$ over $\mathbb{Q}$.
   (a) (3 Points) Show that $E = \mathbb{Q}(\omega, \sqrt[19]{2})$ where $\omega = e^{2\pi i/19}$.
   (b) (4 Points) Show that the size of the Galois group G of E over $\mathbb{Q}$ is $19 \cdot 18$.
   (c) (3 Points) Show that there is an intermediate field L such that $\mathbb{Q} \subset L \subset E$ and L corresponds to a normal subgroup H of G of size 19.
   (d) (4 Points) Prove or disprove: The Galois group G of E over $\mathbb{Q}$ is abelian.
2. (4 Points) Let p be a prime integer and consider $f(x) = x^p - x - 1$ over $\mathbb{F}_p[x]$. Let α be a root of $f(x)$ in the algebraic closure of $\mathbb{F}_p$. Show that $\mathbb{F}_p(\alpha)$ is a Galois extension of $\mathbb{F}_p$. (Hint: if α is a root of $f(x)$ then what about $\alpha + i$ for $1 \leq i \leq p - 1$?)
3. Let $K := \mathbb{Q}(t)$ be the field of rational functions in one variable t over $\mathbb{Q}$. This is the field of fractions of the polynomial ring $R := \mathbb{Q}[t]$ (so every element $q(t) \in K$ can be written as $q(t) = g(t)/h(t)$ with $g, h \in R$). Then K has the structure of an R-module via

$$R \times K \to K, \quad (f(t), q(t)) \mapsto f(t)q(t)$$

   (you do not have to show that this gives K the structure of an R-module).
   (a) (3 points) Let $\varphi \in \mathrm{Hom}_R(K, R)$. By considering $\varphi(t^{-n})$ for positive integers n, show that $\varphi(1) = 0$.
   (b) (1 point) Deduce that $\#\mathrm{Hom}_R(K, R) = 1$.

4. Let $R := \mathbb{Q}[t]$ be the polynomial ring over the rational numbers $\mathbb{Q}$ in one variable t. Then $\mathbb{Q}$ has the structure of an R-module via

$$R \times \mathbb{Q} \to \mathbb{Q}, \quad (f(t), a) \mapsto f(0) \cdot a$$

(you do not have to prove this).

(a) (1 point) Find the torsion submodule $\mathrm{Tor}_R(\mathbb{Q})$.

(b) (5 points) Show that there is an exact sequence of R-modules

$$0 \to tR \to R \to \mathbb{Q} \to 0 .$$

(c) (4 points) Show that no R-submodule of R is isomorphic to $\mathbb{Q}$.

(d) (2 points) Show that the exact sequence in (b) is not split.

(e) (2 points) Is $\mathbb{Q}$ a projective R-module?